How do solve a linear system of equations using a computer?

**5.1 Gaussian Elimination (Naïve)**

If a system is nonsingular:

Ax = b, A in R^nxn, x, b in R^n

Then we can find the unique solution to this system using Gaussian elimination.

Gaussian elimination consists of (1) forward elimination, (2) back substitution

Example:

Identify row 1 as pivot row, and entry 1 as pivot entry

Find multipliers by dividing entry in leading column by pivot entry

Eliminate first variable in the rest of the rows by scaling pivot row and subtracting from original row.

Now this is called a reduced submatrix, which we can repeat the process on, and we end up with

General Algorithm for n x n system

(Using 1-based indexing, i-th row, j-th column)

For k = 1,2,3…, n-1 let k be the pivot row.

We want to do elimination in remaining rows I = k+1, k+2, …, n.

Naïve (Fails to solve several systems):

Ex1) Multipliers do not exist – when the pivot entry equals 0

Ex2) When a small number is introduced

* This leads to subtractive cancellation, unstable computation (we will have to scale first equation with epsilon by a huge multiplier)
* Swapping the two rows, we will have to scale 1 by epsilon, and this leads to a stable computation

Partial pivoting:

* When applying gaussian elimination, you take the row with the largest leading entry in terms of absolute value and make it your pivot row.
  + Still fails for the systems

Scaled partial pivoting:

* Don’t maximize leading coefficients
* Instead maximize the leading coefficient divided by the largest value in row (scale factor)

, where

Ex.

To identify the pivot row, we calculate the scale vector:

So we can choose row 2 or 3 as initial pivot row.

Again we conduct this process

Now you select row #1 as a pivot row

* 1. Problems to Gaussian Elimination

**Problems:**

Accuracy:

* + When representing a matrix, entries of A are changed, can shift to being singular

Complexity:

* + Elimination Stage = O(
  + Substitution Stage =

Example of setting up a linear system that gives you a coefficient of interpolating polynomial:

Gaussian Elimination produces coefficients that aren’t all 1

* Why? Because the Vandermonde matrix is ill-conditioned

Bernstein Basis

Ex.

Instability

Stability has to do with perturbations. In physics you can have a ball staying still at both the bottom and top of the hill, something called steady state, non of the variables we are keeping track of change with time. But a tiny perturbation changes the variables dramatically when the system is unstable, and does not change the system when the system is stable.

Subtractive cancellation is unstable while subtraction is stable:

* Subtraction of machine numbers:

machine precision

* Subtractive Cancellation:

Norms

Def. A norm on a vector space is a non-negative function

that has three properties:

1. (Positive-Definiteness)
2. (Homogeneity)
3. (Triangle Inequality)

Examples:

* Euclidean Norm of a vector of real numbers
* P-norm ( p = 1,2,3,4….)
* Infinity Norm (p =

Norms on matrices

Def. A norm on a matrix is defined to be:

Notes

For the theoretical system , and computer system , where

Is a perturbation of b:

Q: How big is the absolute error in our solution:

A: ||

Q: How big is the relative error in our solution

/

Condition number of A times the relative error in b

* 1. **Factorization of Matrices**

LU decomposition

LU decomposition is about factorizing a matrix into a lower triangular and a upper triangular matrix.

L = unit lower triangular (=1), all diagonal entries are 1, to ensure uniqueness

U = upper triangular matrix

Q: Does an LU factorization exist for every matrix A?

A: An LU-decomposition exists if and only if naïve Gauss elimination works

Q: Why is LU factorization useful?

A: When you have to solve system for multiple RHS vectors: Ax = ,

Solving Ax = b amounts to solving LU x = b

Which amounts to

1. solving Ly = b using forward substitution:
2. solving Ux = y using back substitution:

Note 1: LU decomposition is unique, assuming that it exists

Note 2: There is a second way to get LU decomposition of a matrix

Write out the form of the LU matrices generally, and solve for the coefficients using the definition of matrix multiplication and going across the rows.

Decomposition

L = unit lower triangular

D = diagonal matrix

A necessary condition for decomposition existing is that is symmetric.

One can obtain a decomposition by the decomposition

=

Let

Then

Claim: D is diagonal

Proof:

Apply to the left and to the right

= D

The LHS is the product of lower triangular matrices, and the RHS is an upper triangular matrix, this is possible if and only if those matrices above are diagonal

Cholesky Decomposition

Assumptions A is symmetric, A is positive-semidefinite ().

One can show that

Remark 1: Both assumptions are necessary

<Ax, x> =

Remark 2: A matrix has a cholesky decomposition iff. A is positive-semidefinite and symmetric

Remark 3:A matrix is positive semi-definite if all its eigenvalues are non-negative

Remark 4: Difference between positive-definiteness and positive-semi definiteness is that positive-definite matrices are nonsingular where semi-definite matrices can be singulars.

Used in numerical solutions of PDEs

The conditions number

There is an algorithm for Cholesky decomposition.